MATH 103B – Discussion Worksheet 4
May 4, 2023

Topics: Irreducible polynomials (Judson 17.3); field of fractions (Judson 18.1)

Problem 1. Determine whether the following polynomials are irreducible in the specified ring.

1. $x^2 + x + 1$ in $R[x]$
2. $x^2 + x + 1$ in $\mathbb{F}_3[x]$
3. $x^3 - x + 1$ in $\mathbb{F}_5[x]$
4. $2x^3 + 5x - 3$ in $\mathbb{Q}[x]$
5. $2x^3 - 3$ in $\mathbb{Q}[x]$
6. $3x^4 - 25x^3 + 100x^2 - 75x + 5$ in $\mathbb{Q}[x]$

Problem 2. Prove or disprove: If $f(x)$ is irreducible in $\mathbb{Z}[x]$, then $\overline{f(x)}$ is irreducible in $\mathbb{F}_p[x]$ for every prime number p.

Problem 3. Prove or disprove: If $f(x)$ is reducible in $\mathbb{Z}[x]$, then $\overline{f(x)}$ is reducible in $\mathbb{F}_p[x]$ for every prime number p.

Problem 4. Determine the field of fraction of

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, p \nmid b \right\},$$

and prove your result.

Problem 5. Prove that a field of characteristic p must contain $\mathbb{F}_p$.

Remark 0.1. On your homework, you will prove that a field of characteristic 0 must contain $\mathbb{Q}$.

Problem 6. (Bonus) Recall the ring of formal power series over the real numbers $\mathbb{R}[[x]]$ introduced in class. Show that $f(x) = a_0 + a_1x + a_2x^2 + \ldots \in \mathbb{R}[[x]]$ is a unit if and only if $a_0 \neq 0$. What is the field of fractions of $\mathbb{R}[[x]]$?

Remark 0.2. Using a similar argument, one can show that a power series with coefficients from any ring R is a unit if and only if its constant term is a unit in R.